## IV. *A Supplement to the Third Part of the Paper on the Summation of infinite Series, in the* Philosophical Transactions *for the Year* 1782. *By the Rev.* S. Vince, *M. A.*; *communicated by* Nevil Maskelyne, *D. D. F. R. S. and Astronomer Royal.*

Read November 25, 1784.

THE reasoning in the third part of my paper on the Summation of infinite Series having been misunderstood, I have thought it proper to offer to the Royal Society the following explanation. When I proposed, for example, to sum the series $\frac{1}{2} - \frac{2}{3} + \frac{3}{4} -$ &c. *sine fine*, I wanted to find some quantity which, by its expansion, would produce that series, and that quantity I called its sum; not (as I conceived must have been evident to every one) in the common acceptation of that word, that the more terms we take, the more nearly we should approach to that quantity, and at last arrive nearer to it than by any assignable difference, for there manifestly can be no such quantity; but as being a quantity from which the series must have been deduced by expansion, which quantity I found to be $-\frac{1}{2} + \text{H. L. } 2$. If therefore in the solution of any problem, the conclusion, whose value I want, is expressed by the above series, and which arose from the necessity of expanding some quantity in the preceding part of the operation, surely no one can deny but that I may substitute for it $-\frac{1}{2} + \text{H. L. } 2$. For whatever quantity it was, which by its expansion produced at

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firſt a ſeries, the ſame reduction which, from that ſeries, produced the ſeries $\frac{1}{2} - \frac{2}{3} + \frac{3}{4} -$ &c. muſt alſo have produced $-\frac{1}{2} +$ H. L. 2. from the quantity which was expanded. This value of the ſeries I obtained in the following manner. I ſuppoſed the ſeries $\frac{1}{2} - \frac{2}{3} + \frac{3}{4} -$ &c. to be divided into two parts; the firſt part to contain all the terms till we come to thoſe where the numerators and denominators become both infinitely great, in which caſe every term afterwards may be ſuppoſed to be equal to unity: the ſecond part, therefore, would neceſſarily be (ſuppoſing the firſt part to terminate at an even number of terms) $1 - 1 + 1 - 1 +$ &c. *ſine ſine.* The firſt part, by collecting two terms into one, becomes $-\frac{1}{2 \cdot 3} - \frac{1}{4 \cdot 5} - \frac{1}{6 \cdot 7} -$ &c. which ſeries, as it is continued till the terms become infinitely ſmall, is equal to $-1 +$ H. L. 2. The ſecond part $1 - 1 + 1 -$ &c. has not, taken abſtractedly of its origin, any determinate value (as will be afterwards obſerved), but conſidered as part of the original ſeries it has, for that ſeries muſt have been deduced from the expanſion of the binomial $\overline{1+x}\rceil^{-1}$, or $\frac{1}{1+x}$; and hence, when $x = 1$, $1 - 1 + 1 -$ &c. can in this caſe have come only from $\frac{1}{1+1}$, which, therefore, muſt be ſubſtituted for it; conſequently the two parts together give $-\frac{1}{2} +$ H. L. 2.

Having thus explained the nature of the ſeries which I propoſed to ſum, and the principle upon which the correction depends, I muſt beg leave to acknowledge my obligations to my very worthy and ingenious friend GEORGE ATWOOD, Eſq. F.R.S. who firſt obſerved that the ſeries $1 - 1 + 1 - 1 +$ &c. has no determinate value in the abſtract, as it may be produced by $\frac{1}{1+1+1+\&c.}$ whatever be the number of units in the denomi-

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nator*; and it may alſo be added, that the ſame ſeries ariſes from $\frac{1+1+1+\&c.}{1+1+1+1+\&c.}$, provided the number of units be greater in the denominator than in the numerator. The correction will therefore be different in different circumſtances, and will depend on the nature of the quantity which was at firſt expanded. In the third part of my paper, I applied the correction to thoſe caſes where the original ſeries aroſe from the expanſion of a binomial, where the correction is in general as I there gave it; but as I did not apply my method to any other ſeries, I confeſs that it did not appear to me, that the correction would then be different, which it neceſſarily would had I extended my reaſoning to other caſes. I ſhall therefore add one example to ſhew the method of correction in other inſtances, where the value of the correction will be found to be different, according as we begin to collect at the firſt or ſecond term. Let the ſeries be $\frac{2}{1}-\frac{3}{2}+\frac{5}{4}-\frac{6}{5}+\frac{8}{7}-\&c.$ *ſine fine*, which came originally from $\frac{1}{1+x+x^2}$; now if we begin to collect at the firſt term, the ſeries becomes $\frac{1}{1\cdot 2}+\frac{1}{4\cdot 5}+\&c.$ and for the ſame reaſon as before, the correction, to be added, is $\frac{1}{3}$; but $\frac{1}{1\cdot 2}+\frac{1}{4\cdot 5}+\&c. = \frac{4}{3}$ of a circular arc (A) of $30°$ to the radius $\frac{\sqrt{3}}{2}$; hence the ſum required $=\frac{4}{3}A+\frac{1}{3}$. If we begin to collect at the ſecond term the ſeries becomes $2-\frac{2}{2\cdot 4}-\frac{2}{5\cdot 7}-\&c.$; and the correction to be ſubtracted is $\frac{2}{3}$; for the ſecond part of the original ſeries is now $-1+1-1+1-\&c.$ which was produced by $\frac{1+1}{1+1+1}$; but

* I have been ſince informed by Mr. WALES, F. R. S. that a pupil of his, Mr. POND, made the ſame obſervation.

$2 - \frac{2}{2 \cdot 4} - \frac{2}{5 \cdot 7} - \&c. = 1 + \frac{4}{3}A$; therefore the ſum required = $\frac{1}{3} + \frac{4}{3}A$ as before. In the ſame manner we may apply the correction in all other caſes. Although, therefore, the ſeries $1 - 1 + 1 - 1 + \&c.$ or $-1 + 1 - 1 + 1 - \&c.$ have no determinate value in the abſtract, yet the given ſeries will fix its value by pointing out the quantity from which the ſeries muſt have been originally produced.